MATH 147A SAMPLE MIDTERM

April 28, 2016 The midterm will be 5 problems, some computational, some conceptual. Feel free to bring a reasonable sized paper for notes and formulas.

(1) Compute the curvature of
$$\gamma(t) = (\cos^3(t), \sin^3(t)).$$

(2) Compute the curvature and torsion (if it exists) of
$$\gamma(t) = \left(\frac{4}{5}\cos(t), 1 - \sin(t), -\frac{3}{5}\cos(t)\right)$$
Show that γ parametrizes a circle, find its center, radius and the plane in which it lies.

(3) Show that a unit speed curve with $\kappa(s) > 0$ for each $s \in [a, b]$ is a plane curve if and only if the torsion vanishes everywhere.

(4) State and prove the isoperimetric inequality, assuming Wirtinger's inequality and Green's theorem for area.

(5) Show that a reparametrization by arc-length gives a unit-speed curve.

(6) Derive the Frenet-Serret equation.

(7) Let $\gamma : (a, b) \to \mathbb{R}^2$ be a regular plane curve and let $a \in \mathbb{R}^2$ such that $\gamma(t) \neq a$ for all t. If there exists a $t_0 \in (a, b)$ such that
$$\|\gamma(t) - a\| \geq \|\gamma(t_0) - a\|$$
for all $t \in (a, b)$, show that the straight line joining the point a with $\gamma(t_0)$ is the normal line of γ at t_0. The same is true if we reverse the inequality. Draw a situation that illustrates both cases.

(8) Let $\gamma : (a, b) \to \mathbb{R}^2$ be a regular plane curve and let $[\alpha, \beta] \subset (a, b)$ be such that $\gamma(\alpha) \neq \gamma(\beta)$. Prove that there exists some $t_0 \in (\alpha, \beta)$ such that the tangent line of γ at t_0 is parallel to the segment of the straight line joining $\gamma(\alpha)$ with $\gamma(\beta)$. This is a curve version of mean value theorem. Hint: Consider $f(t) = \det(\gamma(t), \gamma(\beta) - \gamma(\alpha))$.

(9) Prove that a unit speed curve $\gamma : (a, b) \to \mathbb{R}^2$ is an arc of a circle if and only if all its normal lines pass through a given point.

(10) Let $\gamma : (a, b) \to \mathbb{R}^3$ be a unit speed curve with positive curvature. If $\|\gamma(s)\| = 1$ for all s, i.e. γ is a curve on a sphere, and it has constant torsion τ, prove that there exists $b, c \in \mathbb{R}$ such that
$$\kappa(s) = \frac{1}{b\cos(\tau s) + c\sin(\tau s)}.$$

(11) Let γ be a unit speed curve in $\mathbb{R}^3$ with constant curvature and zero torsion. Show that γ is a parametrization of a circle.

Solutions

1. Book problem 2.1.1
2. Book problem 2.3.1
3. Book Prop 2.3.3
4. Book Theorem 3.2.2
5. Let $\gamma(t)$ be a curve and let $s(t) = \int_0^t \|\dot\gamma(u)\| du$ be the arc length. Then
$$\frac{d}{ds}\gamma(t) = \dot\gamma \frac{dt}{ds} = \frac{\dot\gamma}{\frac{ds}{dt}} = \frac{\dot\gamma}{\|\dot\gamma\|}.$$

Hence $\|\frac{d\gamma}{ds}\| = 1$ (as long as γ is regular).

6. See paragraph before Theorem 2.3.4

7. Let $f(t) = \|\gamma(t) - a\|^2$. Then $t = t_0$ is a minimum therefore

$$0 = f'(t_0) = 2\langle \gamma'(t_0), \gamma(t_0) - a\rangle.$$

The same proof will work for a maximum.

8. We have $f(a) = f(b) = 0$. Therefore, by mean value theorem, there is a point such that

$$0 = f'(t_0) = \det(\gamma'(t_0), \gamma(\beta) - \gamma(\alpha))$$

This shows that $\gamma'(t_0)$ and $\gamma(\beta) - \gamma(\alpha)$ are linearly dependent and for $\mathbb{R}^2$, they are parallel.

9. Suppose γ is an arc of a circle. If we center at the origin, then the position vector is the normal vector, hence passes through the center. Conversely, Suppose γ is a unit speed curve such that all its normal lines pass through a given point, say a. Consider $f(t) = \|\gamma(t) - a\|^2$. Then

$$f'(t) = 2\langle \gamma'(t), \gamma(t) - a\rangle.$$

The vector $\gamma(t) - a$ is a normal line by assumption and $\gamma'(t)$ is a tangent vector, hence they are perpendicular so $f'(t) = 0$ for all t where the assumption holds. Hence $f(t)$ is a constant, i.e. $\|\gamma(t) - a\| = R$ for some R.

10. Differentiating $\|\gamma(s)\|^2 = 1$ twice, we get

$$\begin{aligned} &\langle \gamma'(s), \gamma(s)\rangle = 0 \\ &\langle \gamma''(s), \gamma(s)\rangle + \langle \gamma'(s), \gamma'(s)\rangle = 0. \end{aligned}$$

Using the fact that $\gamma' = T$, $\|\gamma'(s)\| = 1$ and $\gamma''(s) = \dot{T} = \kappa N$,

$$\langle N(s), \gamma(s)\rangle = -\frac{1}{\kappa}.$$

Taking the derivative once more, we have

$$-\left(\frac{1}{\kappa}\right)' = \langle \dot{N}, \gamma\rangle + \langle N, \dot{\gamma}\rangle = \langle \dot{N}, \gamma\rangle$$

since $\dot{\gamma}$ and N are perpendicular. From the Frenet-Serret equations,

$$\langle \dot{N}, \gamma\rangle = -K\langle T, \gamma\rangle + \tau\langle B, \gamma\rangle = -K\langle \dot{\gamma}, \gamma\rangle + \tau\langle B, \gamma\rangle = \tau\langle B, \gamma\rangle$$

where we used the fact that γ lies on a sphere so $\langle \dot{\gamma}, \gamma\rangle = 0$. Finally taking the derivative once more

$$-\left(\frac{1}{\kappa}\right)'' = \tau\langle \dot{B}, \gamma\rangle + \tau\langle B, \dot{\gamma}\rangle = \tau\langle \dot{B}, \gamma\rangle$$

and use the Frenet-Serret equations once more to get

$$-\left(\frac{1}{\kappa}\right)'' = -\tau^2\langle N, \gamma\rangle = -\tau^2\left(\frac{1}{\kappa}\right).$$

Hence we want to solve an ODE of the form $y'' + \tau^2 y = 0$. It has a solution

$$y(s) = b\cos(\tau s) + c\sin(\tau s)$$

for some constants b and c. Since $y = \frac{1}{\kappa}$, we obtain the result.

11. Book Prop 2.3.5